POSITION-BASED SIMULATION METHODS IN COMPUTER GRAPHICS

Jan Bender, Matthias Müller, Miles Macklin



Eurographics 2015 The 36th Annual Conference of the European Association for Computer Graphics

Topics

- Introduction
  Jan Bender
- Basic Position-Based Dynamics
  Matthias Müller
- Advanced Position-Based Dynamics
  Jan Bender
- Implementation & Analysis
  Miles Macklin
- Conclusion
  Jan Bender

Introduction

- Motivation
- Background

Motivation

- Physical simulation is a well studied problem in the computational sciences.
- Many well established methods have been adopted in computer graphics:
  - Finite Element Method (FEM) [OH99]
  - Finite Differences Method [TPBF87b]
  - Finite Volume Method [TBHF03]
  - Boundary Element Method [JP99]
  - Particle-based approaches [DSB99, THMG04]

Motivation

- Main applications of physically based simulation in computer graphics:
  - special effects in movies and commercials
  - computer games
  - other interactive systems
- Speed, stability and controllability are the most important factors.
- Only visual plausibility is required.

Position-Based Methods

- provide a high level of control
- are stable even in combination with explicit time integration
- are very fast
- are simple to implement
- provide visually plausible results

Position-Based Methods

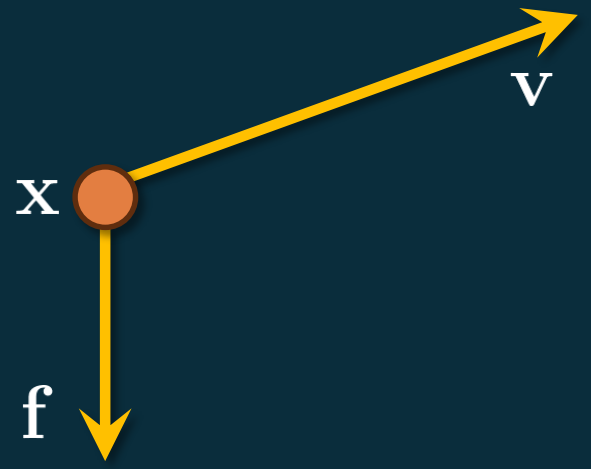


Introduction

- Motivation
- Background

Particle

- 3 translational degrees of freedom (DOF)
- Physical properties:
  - Mass $m \in \mathbb{R}$
  - Position $\mathbf{x} \in \mathbb{R}^3$
  - Velocity $\mathbf{v} \in \mathbb{R}^3$



Equations of Motion: Particle

- Newton's second law:

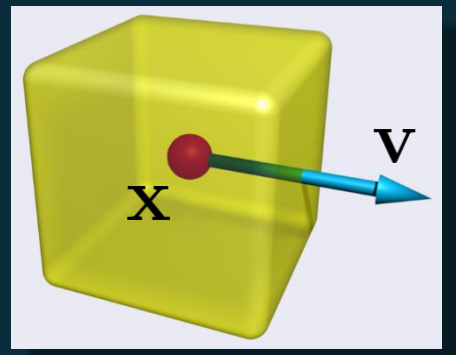
$$\dot{\mathbf{v}} = \frac{1}{m}\mathbf{f}$$

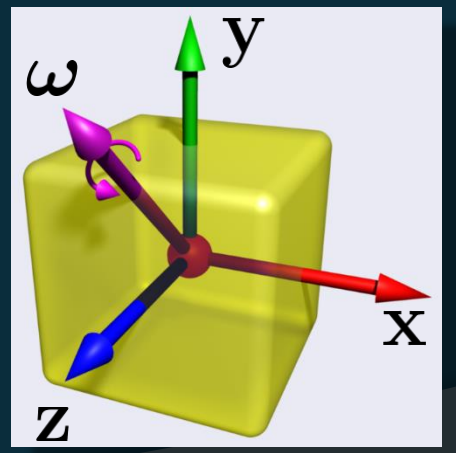
- Velocity kinematic relationship:

$$\dot{\mathbf{x}} = \mathbf{v}$$

Rigid Body

- 6 DOFs
- Translational properties:
  - Mass $m \in \mathbb{R}$
  - Position $\mathbf{x} \in \mathbb{R}^3$
  - Velocity $\mathbf{v} \in \mathbb{R}^3$
- Rotational properties
  - Inertia tensor $\mathbf{I} \in \mathbb{R}^{3\times3}$
  - Orientation $\mathbf{q} \in \mathbb{H}$
  - Angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$





Equations of Motion: Rigid Body

- Euler's equations:

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1}\left(\boldsymbol{\tau} - (\boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}))\right)$$

- Velocity kinematic relationship:

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{2}\tilde{\boldsymbol{\omega}}\mathbf{q} \\ \tilde{\boldsymbol{\omega}} &= [0, \omega^x, \omega^y, \omega^z] \end{aligned}$$

Symplectic Euler Integration

- Translational part:

$$\begin{aligned} \mathbf{v} &\leftarrow \mathbf{v} + \triangle t \frac{1}{m} \mathbf{f} \\ \mathbf{x} &\leftarrow \mathbf{x} + \triangle t \mathbf{v} \end{aligned}$$

- Rotational part:

$$\begin{aligned} \boldsymbol{\omega} &\leftarrow \boldsymbol{\omega} + \triangle t \, \mathbf{I}^{-1} \left(\boldsymbol{\tau} - (\boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})) \right) \\ \mathbf{q} &\leftarrow \mathbf{q} + \triangle t \, \frac{1}{2} \tilde{\boldsymbol{\omega}} \mathbf{q} \end{aligned}$$

Constraints

- Bilateral (holonomic) constraint:

$$C(\mathbf{x}_{i_1}, \mathbf{q}_{i_1}, \ldots, \mathbf{x}_{i_{n_j}}, \mathbf{q}_{i_{n_j}}) = 0$$

- Unilateral constraint:

$$C(\mathbf{x}_{i_1}, \mathbf{q}_{i_1}, \ldots, \mathbf{x}_{i_{n_j}}, \mathbf{q}_{i_{n_j}}) \geq 0,$$

Topics

- Introduction
  Jan Bender
- Basic Position-Based Dynamics
  Matthias Müller
- Advanced Position-Based Dynamics
  Jan Bender
- Implementation & Analysis
  Miles Macklin
- Conclusion
  Jan Bender

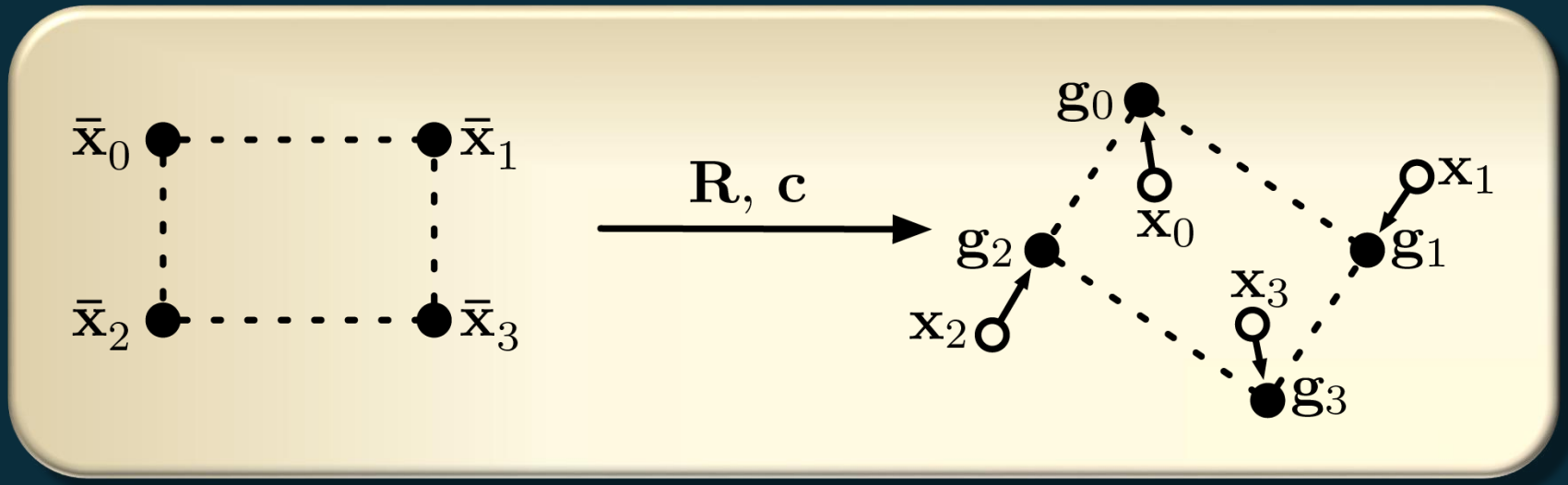
Topics

- Introduction
  Jan Bender
- Basic Position-Based Dynamics
  Matthias Müller
- Advanced Position-Based Dynamics
  Jan Bender
- Implementation & Analysis
  Miles Macklin
- Conclusion
  Jan Bender

Advanced Position-Based Dynamics

- Shape Matching
- Continuous Materials
- Rigid Body Dynamics

Shape Matching



- Initial vertex positions: $\bar{\mathbf{x}}_i$
- Deformed configuration: $\mathbf{x}_i$
- Goal positions: $\mathbf{g}_i$

Optimal Translation

- The optimal translation vectors are given by the centers of mass:

$$\bar{\mathbf{c}} = \frac{1}{M}\sum_i m_i \bar{\mathbf{x}}_i$$

$$\mathbf{c} = \frac{1}{M}\sum_i m_i \mathbf{x}_i$$

$$M = \sum_i m_i$$

Optimal Rotation

- The optimal linear transformation:

$$\mathbf{A} = \left(\sum_i m_i \mathbf{r}_i \bar{\mathbf{r}}_i^T \right) \left(\sum_i m_i \bar{\mathbf{r}}_i \bar{\mathbf{r}}_i^T \right)^{-1} = \mathbf{A}_r \mathbf{A}_s$$

with

$$\begin{aligned} \mathbf{r}_i &= \mathbf{x}_i - \mathbf{c} \\ \bar{\mathbf{r}}_i &= \bar{\mathbf{x}}_i - \bar{\mathbf{c}} \end{aligned}$$

Optimal Rotation

- Optimal rotation is rotational part of $\mathbf{A} = \mathbf{A}_r \mathbf{A}_s$.
- $\mathbf{A}_s$ is symmetric $\rightarrow$ it contains no rotation.
- Extract rotational part of $\mathbf{A}_r$ by polar decomposition:

$$\mathbf{A}_r = \mathbf{RS}$$

Goal Positions

- Finally, the goal positions are determined by

$$\mathbf{g}_i = \mathbf{T} \begin{bmatrix} \bar{\mathbf{x}}_i \\ 1 \end{bmatrix}$$

with the transformation matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & (\mathbf{c} - \mathbf{R}\bar{\mathbf{c}}) \end{bmatrix}$$

Time Integration

- Modified integration scheme:

$$\begin{aligned} \mathbf{v}_i &\leftarrow \mathbf{v}_i + \Delta t \frac{1}{m_i} \mathbf{f}_{\text{ext}} + \alpha \frac{\mathbf{g}_i - \mathbf{x}_i}{\Delta t} \\ \mathbf{x}_i &\leftarrow \mathbf{x}_i + \Delta t \mathbf{v}_i \end{aligned}$$

with user-defined stiffness $\alpha \in [0, 1]$

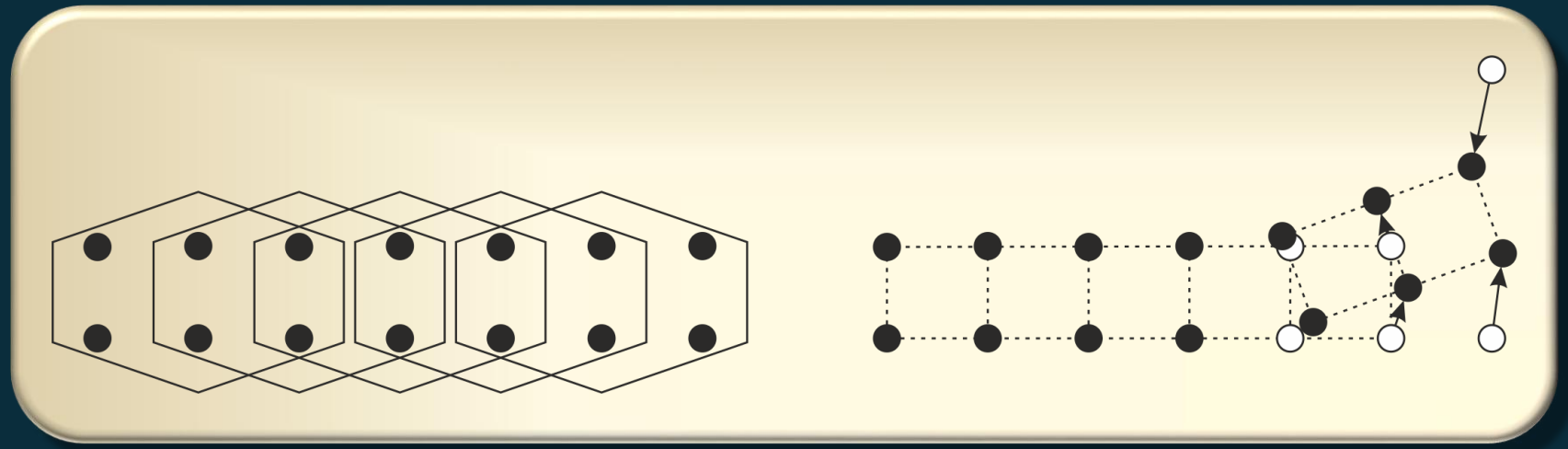
Stability Test

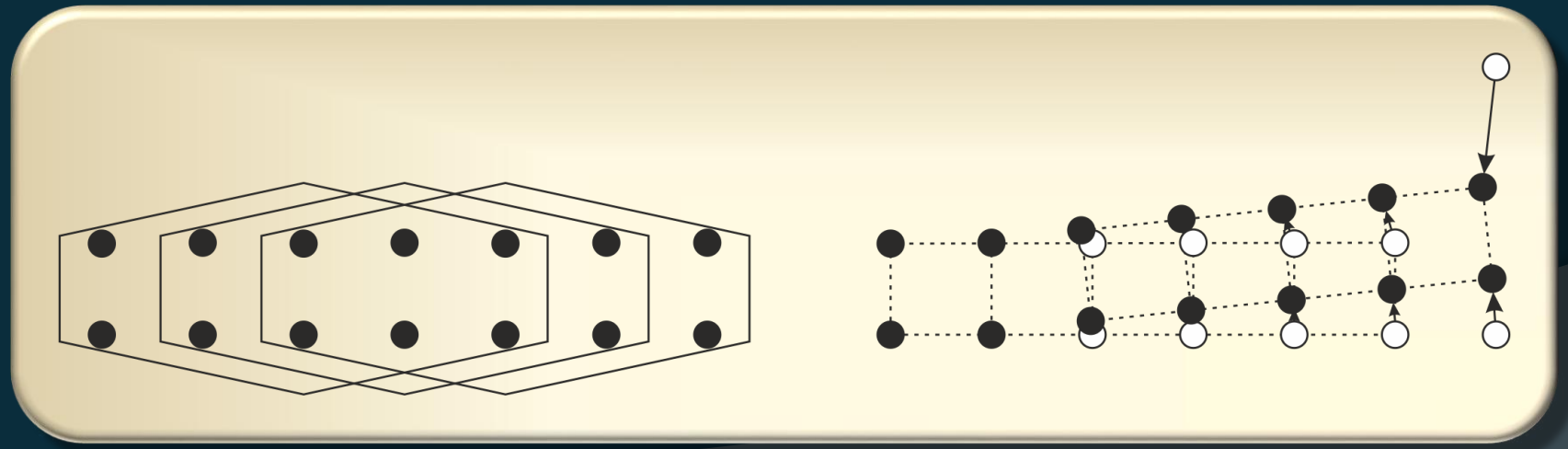


Region-Based Shape Matching

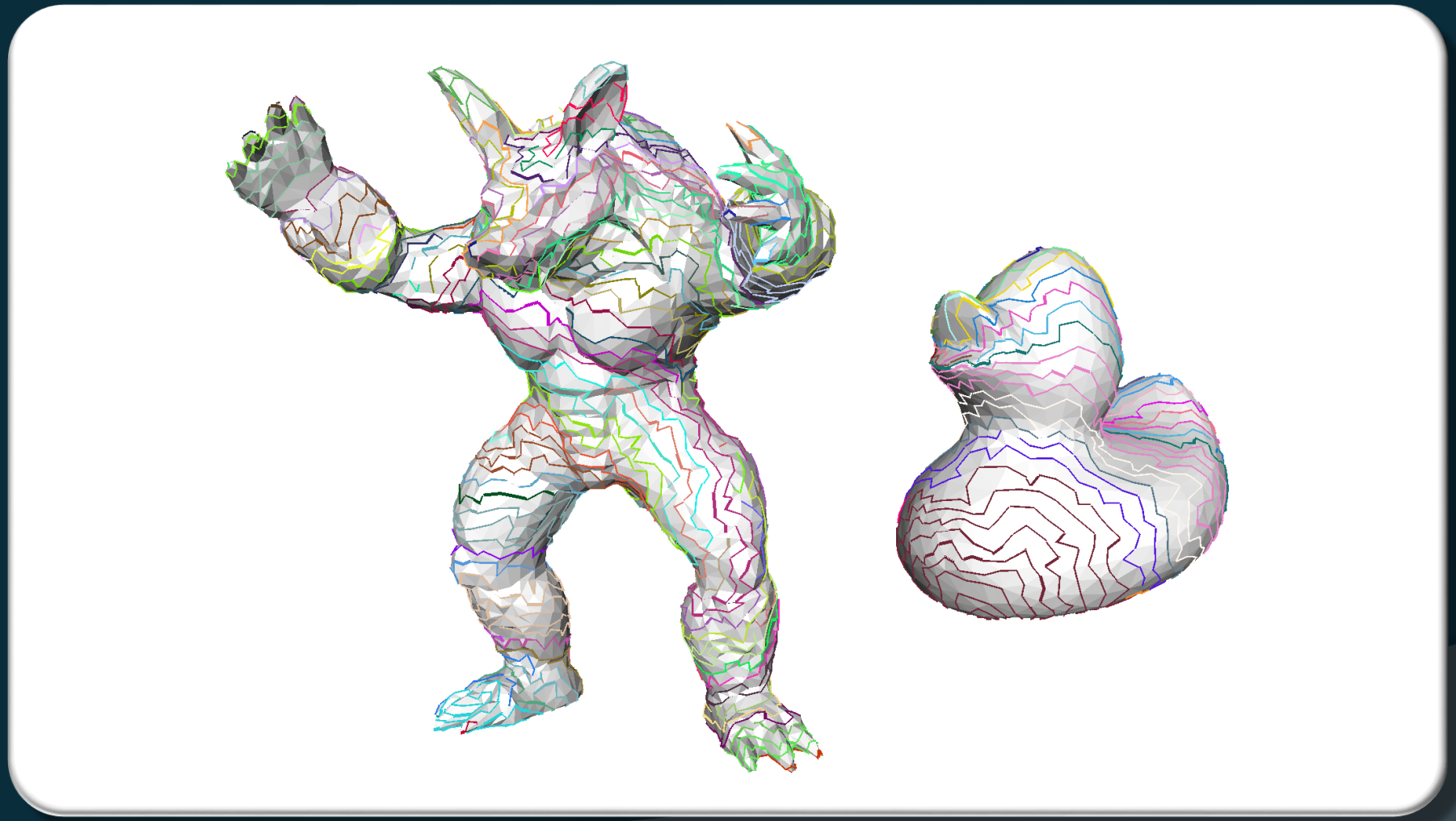
- Shape matching allows only small deviations from the initial shape.
- Large deformations can be simulated by performing shape matching on several overlapping regions.
- Each particle is part of multiple regions.

Region-Based Shape Matching

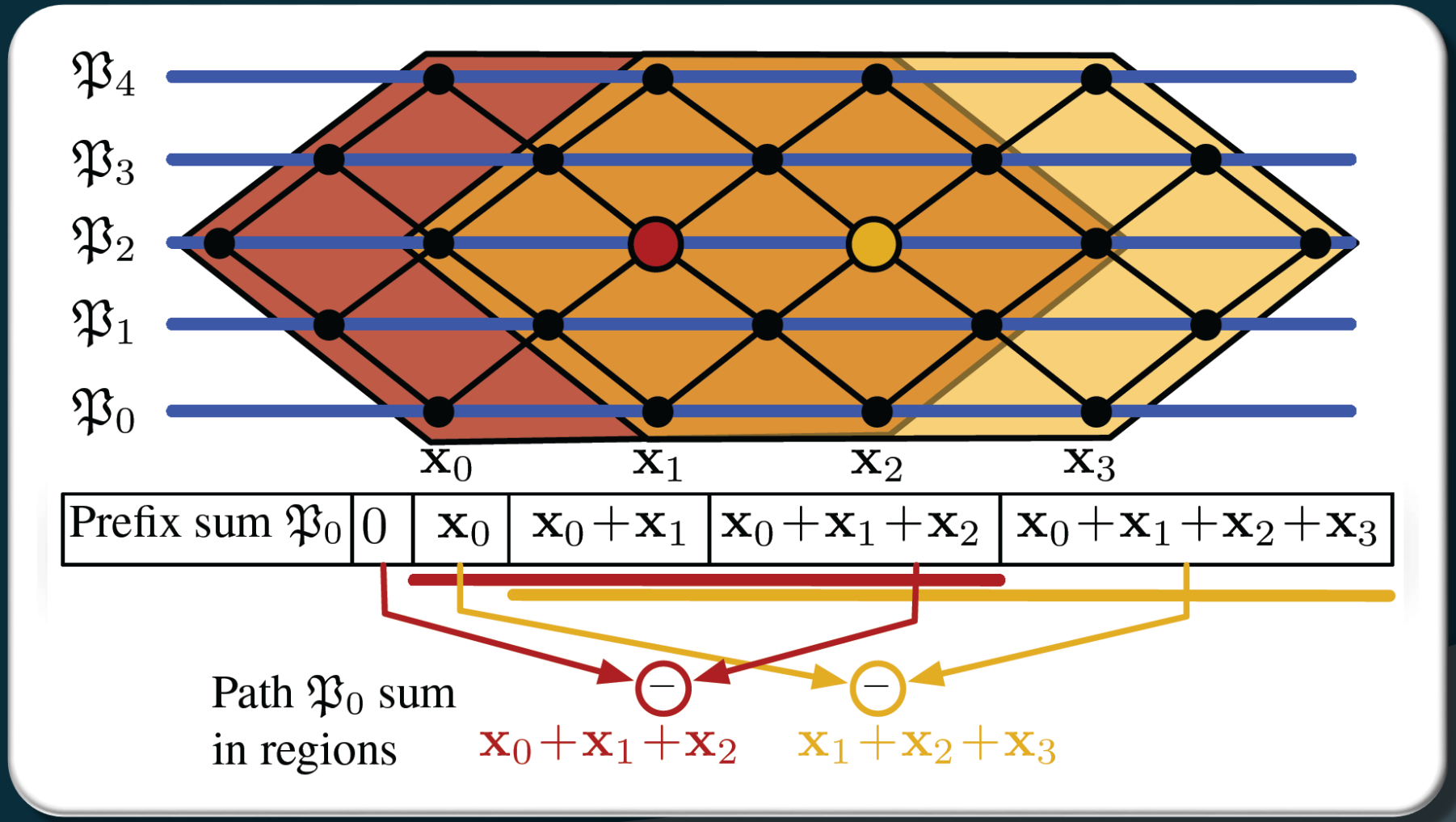




Fast Summation



Fast Summation



Results



3.54ms
21280 particles



2.34ms
7640 particles



4.69ms
32442 particles

Times per step
on a GTX 470

Oriented Particles

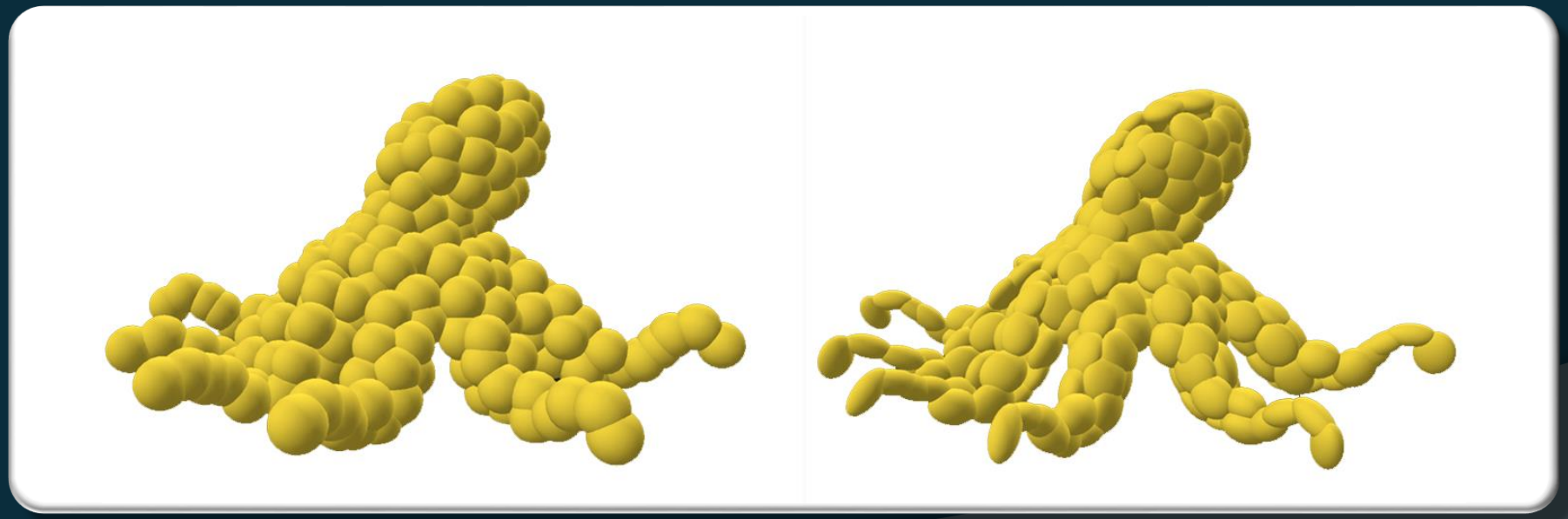
- For almost co-linear or co-planar particle configurations the polar decomposition tends to be numerically unstable.

- Solution:

  Add orientation information to particles

Oriented Particles

- Orientation information can also be used
  - to position anisotropic collision shapes like ellipsoids
  - for robust skinning of the visual mesh



Moment Matrix

- Spherical particle with radius r:

$$\mathbf{A}_i^{\text{sphere}} = \frac{1}{5} m r^2 \mathbf{R}$$

- Ellipsoid with radii a, b and c:

$$\mathbf{A}_i^{\text{ellipsoid}} = \frac{1}{5} m \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} \mathbf{R}$$

- $\mathbf{R}$ is orthonormal $\rightarrow$ $\mathbf{A}_i$ has full rank

Shape Matching with Oriented Particles

- Determine optimal translational vectors:

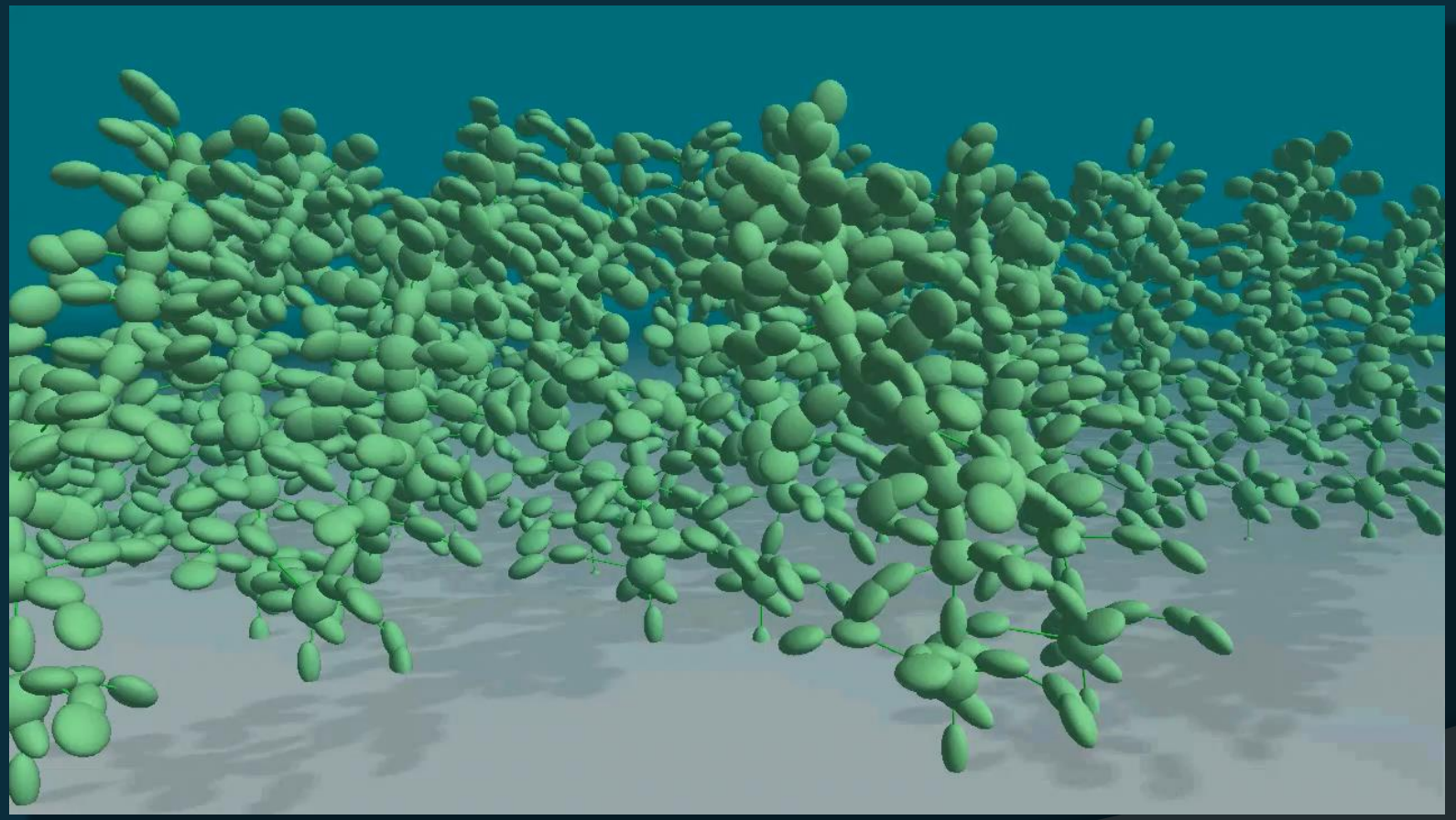
$$\bar{\mathbf{c}} = \frac{\sum_i m_i \bar{\mathbf{x}}_i}{\sum_i m_i}, \qquad \mathbf{c} = \frac{\sum_i m_i \mathbf{x}_i}{\sum_i m_i}$$

- Optimal rotation is rotational part of

$$\mathbf{A} = \sum_i \left(\mathbf{A}_i + m_i (\mathbf{x}_i - \mathbf{c})(\bar{\mathbf{x}}_i - \bar{\mathbf{c}}^T) \right)$$

where $\mathbf{A}_i$ are the moment matrices.

Oriented Particles

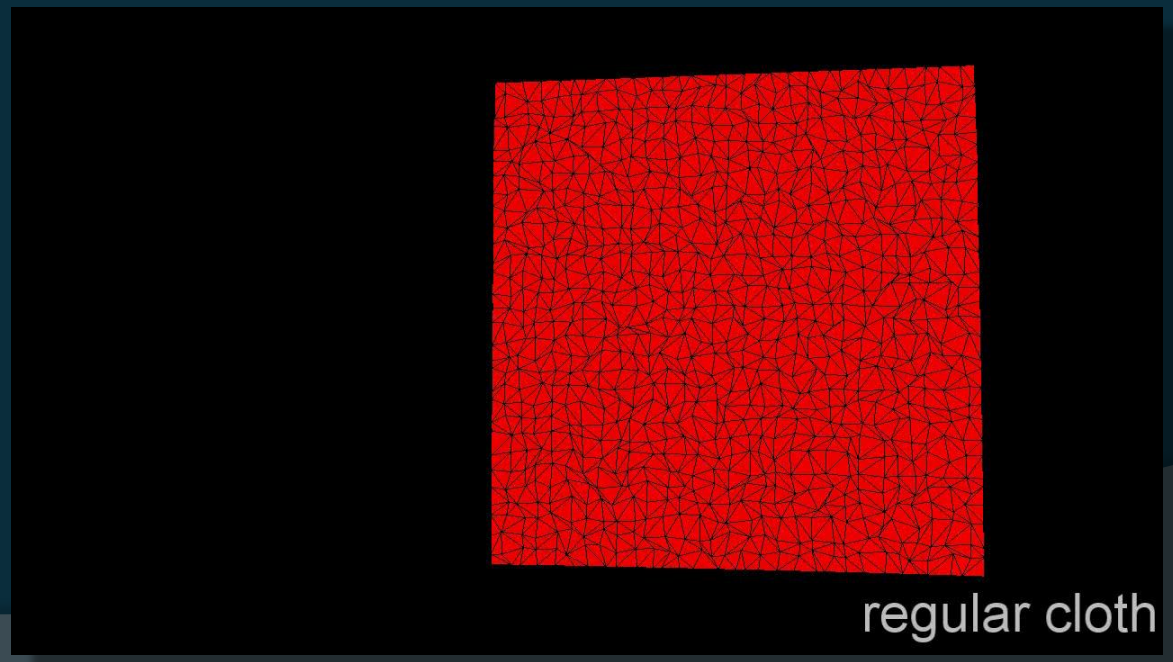


Advanced Position-Based Dynamics

- Shape Matching
- Continuous Materials
- Rigid Body Dynamics

Continuous Materials

- Idea: use a continuum-based formulation in PBD
- Enables control of principal strains on irregular mesh
- Allows complex physical effects like lateral contraction, anisotropy or elastoplasticity



Deformation

- Deformation

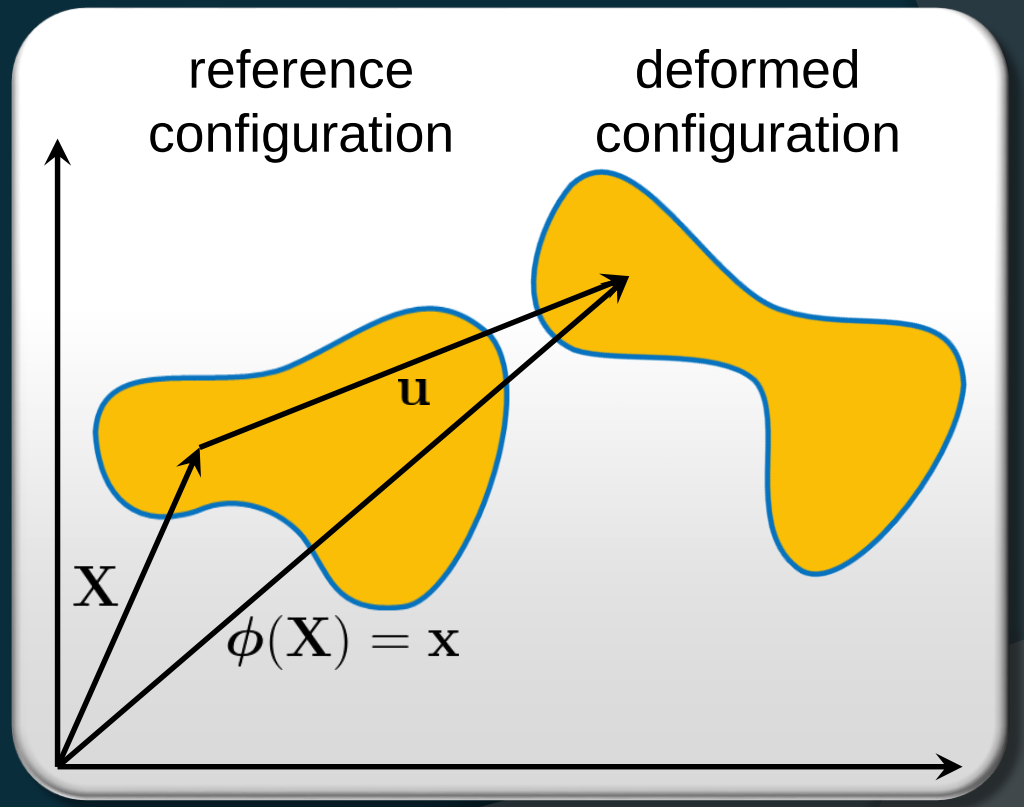
$$\phi(\mathbf{X}) = \mathbf{X} + \mathbf{u} = \mathbf{x}$$

- Deformation gradient

$$\mathbf{F} = \partial\phi(\mathbf{X})/\partial\mathbf{X}$$

- Non-linear Green strain tensor

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$$



Strain Energy

- Hooke's generalized law

$$\mathbf{S} = \mathbf{C}\boldsymbol{\epsilon}$$

- Strain energy density field

$$\Psi_s = \frac{1}{2}\boldsymbol{\epsilon} : \mathbf{S} = \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\epsilon}^T\mathbf{S}\right]$$

Strain Energy

- Energy

$$E_s = \int_\Omega \Psi_s d\mathbf{X}$$

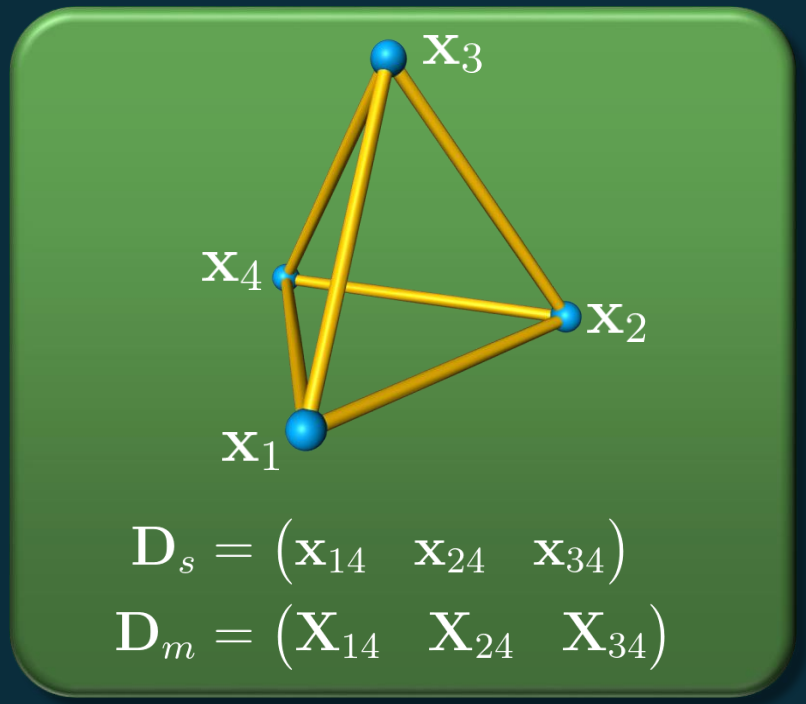
- Energy gradient

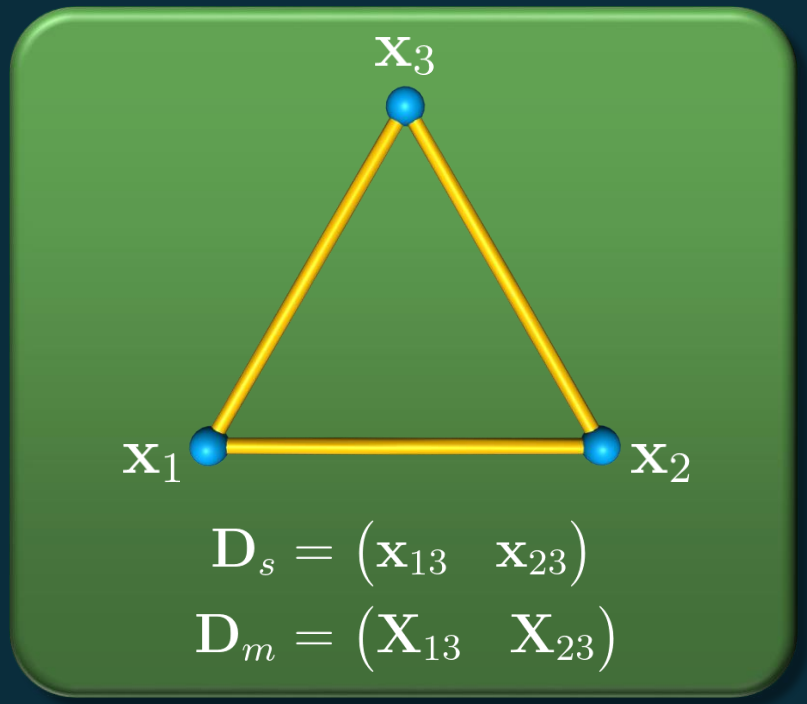
$$\frac{\partial E_s}{\partial \mathbf{x}} = \int_\Omega \frac{\partial \Psi_s}{\partial \mathbf{x}} d\mathbf{X} = \int_\Omega \mathbf{P}(\mathbf{F}) \frac{\partial \mathbf{F}}{\partial \mathbf{x}} d\mathbf{X}$$

- First Piola-Kirchhoff stress tensor

$$\mathbf{P}(\mathbf{F}) = \mathbf{F}\mathbf{C}\boldsymbol{\epsilon}$$

Discretization



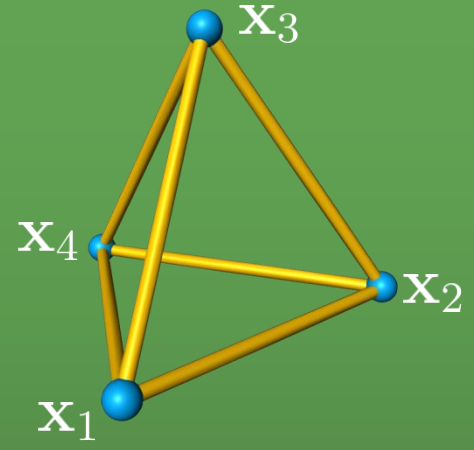


Deformation gradient:

$$\mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{-1}$$

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \quad \mathbf{X}_{ij} = \mathbf{X}_i - \mathbf{X}_j$$

Elastic Energy

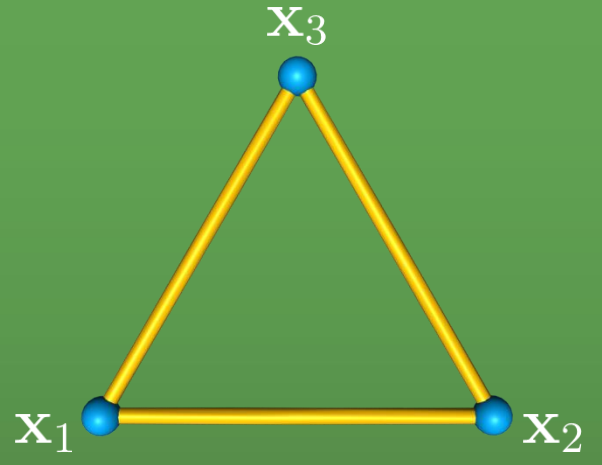


Energy: $E_s = V\Psi_s(\mathbf{F})$

Gradient:

$$\left[\frac{\partial E_s}{\partial \mathbf{x}_1} \; \frac{\partial E_s}{\partial \mathbf{x}_2} \; \frac{\partial E_s}{\partial \mathbf{x}_3}\right] = V\mathbf{P}(\mathbf{F})\mathbf{D}_m^{-T}$$

$$\frac{\partial E_s}{\partial \mathbf{x}_4} = -\sum_{i=1}^{3} \frac{\partial E_s}{\partial \mathbf{x}_i}$$



Energy: $E_s = A\Psi_s(\mathbf{F})$

Gradient:

$$\left[\frac{\partial E_s}{\partial \mathbf{x}_1} \; \frac{\partial E_s}{\partial \mathbf{x}_2}\right] = A\mathbf{P}(\mathbf{F})\mathbf{D}_m^{-T}$$

$$\frac{\partial E_s}{\partial \mathbf{x}_3} = -\sum_{i=1}^{2} \frac{\partial E_s}{\partial \mathbf{x}_i}$$

Position-Based Simulation

- Energy constraint:

$$C = E_s$$

- Constraint gradient:

$$\frac{\partial C}{\partial \mathbf{x}} = \frac{\partial E_s}{\partial \mathbf{x}}$$

Plastic Deformation

- Elastic and plastic strain:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{\mathrm{e}} + \boldsymbol{\epsilon}_{\mathrm{p}}$$

- Update of plastic part:

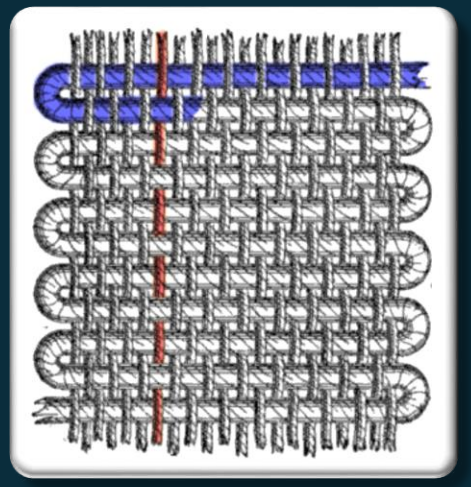
$$\boldsymbol{\epsilon}_{\mathrm{p}} := \begin{cases} \boldsymbol{\epsilon}_{\mathrm{p}} + \triangle t \gamma_{\mathrm{creep}} \boldsymbol{\epsilon}_{\mathrm{e}} & \text{if } \gamma_{\mathrm{yield}} < \|\boldsymbol{\epsilon}_{\mathrm{e}}\| < \gamma_{\mathrm{max}} \\ \gamma_{\mathrm{max}} \dfrac{\boldsymbol{\epsilon}_{\mathrm{p}}}{\|\boldsymbol{\epsilon}_{\mathrm{p}}\|} & \text{if } \|\boldsymbol{\epsilon}_{\mathrm{e}}\| \geq \gamma_{\mathrm{max}} \end{cases}$$



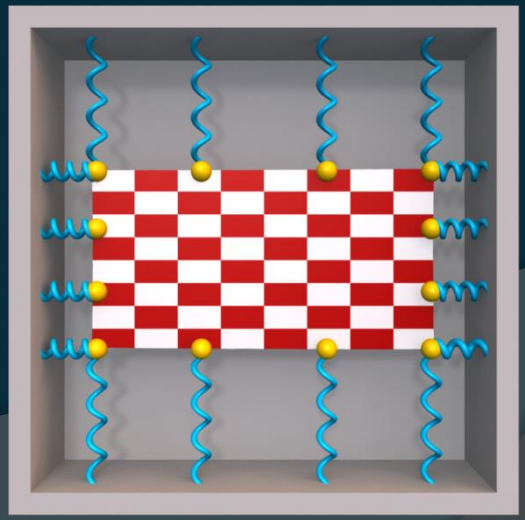
Anisotropy

- Example:
  - Orthotropic material behavior of cloth
- Elasticity tensor:

$$\mathbf{C} = \begin{pmatrix} \frac{k_{\text{weft}}}{1-\nu_{\text{weft}}\nu_{\text{warp}}} & \frac{k_{\text{weft}}\nu_{\text{warp}}}{1-\nu_{\text{weft}}\nu_{\text{warp}}} & 0 \\ \frac{k_{\text{warp}}\nu_{\text{weft}}}{1-\nu_{\text{weft}}\nu_{\text{warp}}} & \frac{k_{\text{warp}}}{1-\nu_{\text{weft}}\nu_{\text{warp}}} & 0 \\ 0 & 0 & k_{\text{shear}} \end{pmatrix}$$



Source: Wikipedia



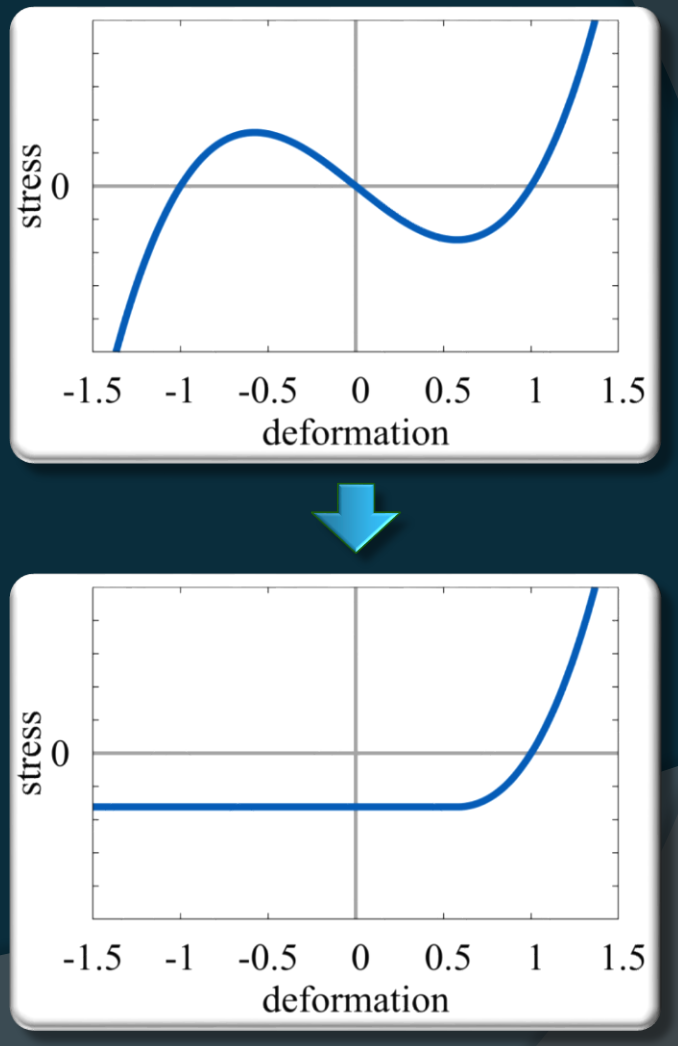
Inverted Elements

- Determine diagonalized deformation gradient $\hat{\mathbf{F}}$ by singular value decomposition:

$$\mathbf{F} = \mathbf{U}\hat{\mathbf{F}}\mathbf{V}^T$$

- Modify diagonalized stress tensor $\hat{\mathbf{P}} = \mathbf{P}(\hat{\mathbf{F}})$:

$$\mathbf{P}' = \mathbf{U}\hat{\mathbf{P}}'\mathbf{V}^T$$

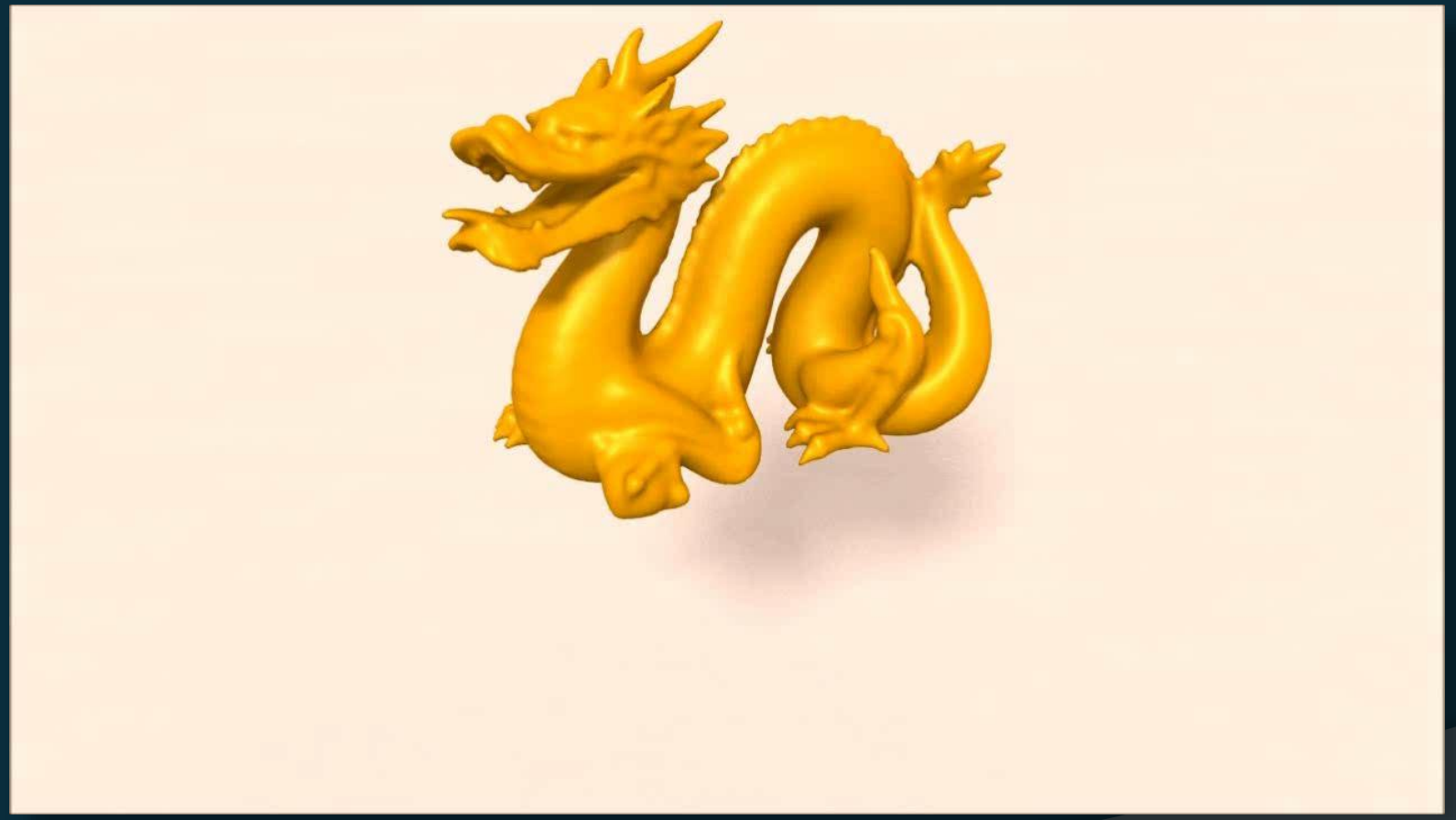


Performance



Model	# vert.	# elem.	elasticity	bending	total
Armadillos	118000	371700	152.41 ms	-	152.41 ms
Cloth	40401	80000	4.11 ms	2.62 ms	6.73 ms

Stability



Effects



Strain Based Dynamics

- Idea: directly define constraints for Green strain tensor $\mathbf{F}^T\mathbf{F} - \mathbf{I}$
  - Stretch constraints:

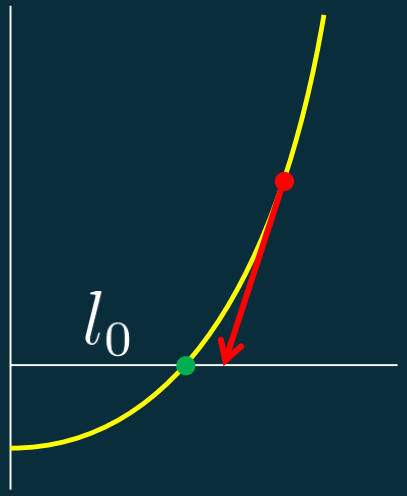
$$C(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbf{S}_{ii} - 1, \qquad \mathbf{S} = \mathbf{F}^T\mathbf{F}$$

  - Shear constraints:

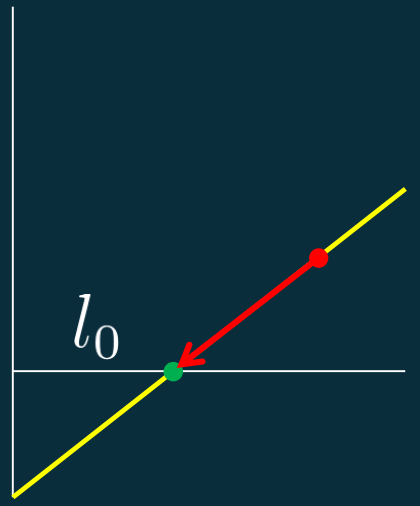
$$C(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbf{S}_{ij}, \qquad i < j$$

Linear vs. Quadratic Strain

$$C(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|^2 - l_0^2$$



$$C(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2| - l_0$$



- Linearized stretch constraint can be solved in one step:

$$C(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \sqrt{\mathbf{S}_{ii}} - 1$$

Decoupled Shear Strain

- Current shear constraint is stretch dependent:

$$C(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbf{f}_i \cdot \mathbf{f}_j, \qquad \text{where } \mathbf{F} = [\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3]$$

- Stretch independent:

$$C(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{\mathbf{f}_i \cdot \mathbf{f}_j}{\|\mathbf{f}_i\| \, \|\mathbf{f}_j\|}$$

Results

Clothing

- Tight clothing in bind pose causes problems in significantly different poses



Skin Sliding

- Low tangential shear resistance used for skin sliding



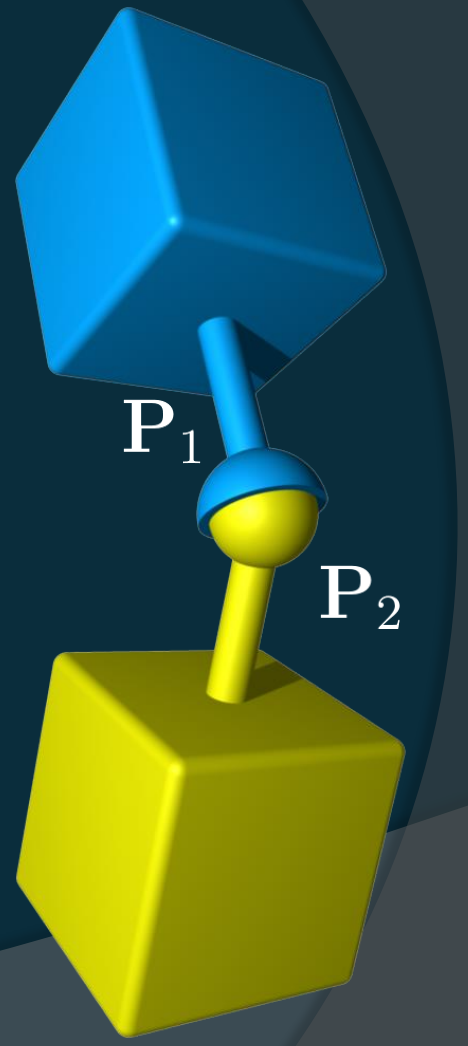
Advanced Position-Based Dynamics

- Shape Matching
- Continuous Materials
- Rigid Body Dynamics

Connectors

- Particle: $\mathbf{P}(\mathbf{x}) = \mathbf{x}$
- Rigid body: $\mathbf{P}(\mathbf{x}, \boldsymbol{\vartheta}) = \mathbf{x} + \mathbf{R}(\boldsymbol{\vartheta})\mathbf{P}_{\text{local}}$
- Rotation is parameterized by exponential map $\boldsymbol{\vartheta}$
- Example: ball joint

$$\mathbf{C}(\mathbf{P}_1, \mathbf{P}_2) = \mathbf{P}_1 - \mathbf{P}_2 = \mathbf{0}$$



Gradients

- Split gradient into a constraint part and connector part
  - Particle:

$$\nabla \mathbf{C} = \underbrace{\frac{\partial \mathbf{C}(\mathbf{P})}{\partial \mathbf{P}}}_{\substack{\text{constraint} \\ \text{specific part}}} \cdot \underbrace{\frac{\partial \mathbf{P}}{\partial \mathbf{x}}}_{\substack{\text{connector} \\ \text{specific part}}} = \frac{\partial \mathbf{C}(\mathbf{P})}{\partial \mathbf{P}}$$

  - Rigid Body:

$$\nabla \mathbf{C} = \underbrace{\frac{\partial \mathbf{C}(\mathbf{P})}{\partial \mathbf{P}}}_{\substack{\text{constraint} \\ \text{specific part}}} \cdot \underbrace{\begin{pmatrix} \frac{\partial \mathbf{P}}{\partial \mathbf{x}} & \frac{\partial \mathbf{P}}{\partial \boldsymbol{\vartheta}} \end{pmatrix}^T}_{\substack{\text{connector} \\ \text{specific part}}}$$

Position-Based Solver

- Linearization of constraint (rigid bodies):

$$\mathbf{C}(\mathbf{x} + \triangle\mathbf{x}, \boldsymbol{\vartheta} + \triangle\boldsymbol{\vartheta}) \approx \mathbf{C}(\mathbf{x}, \boldsymbol{\vartheta}) + \mathbf{J}(\mathbf{x}, \boldsymbol{\vartheta}) \left(\triangle\mathbf{x}^T, \triangle\boldsymbol{\vartheta}^T\right)^T$$

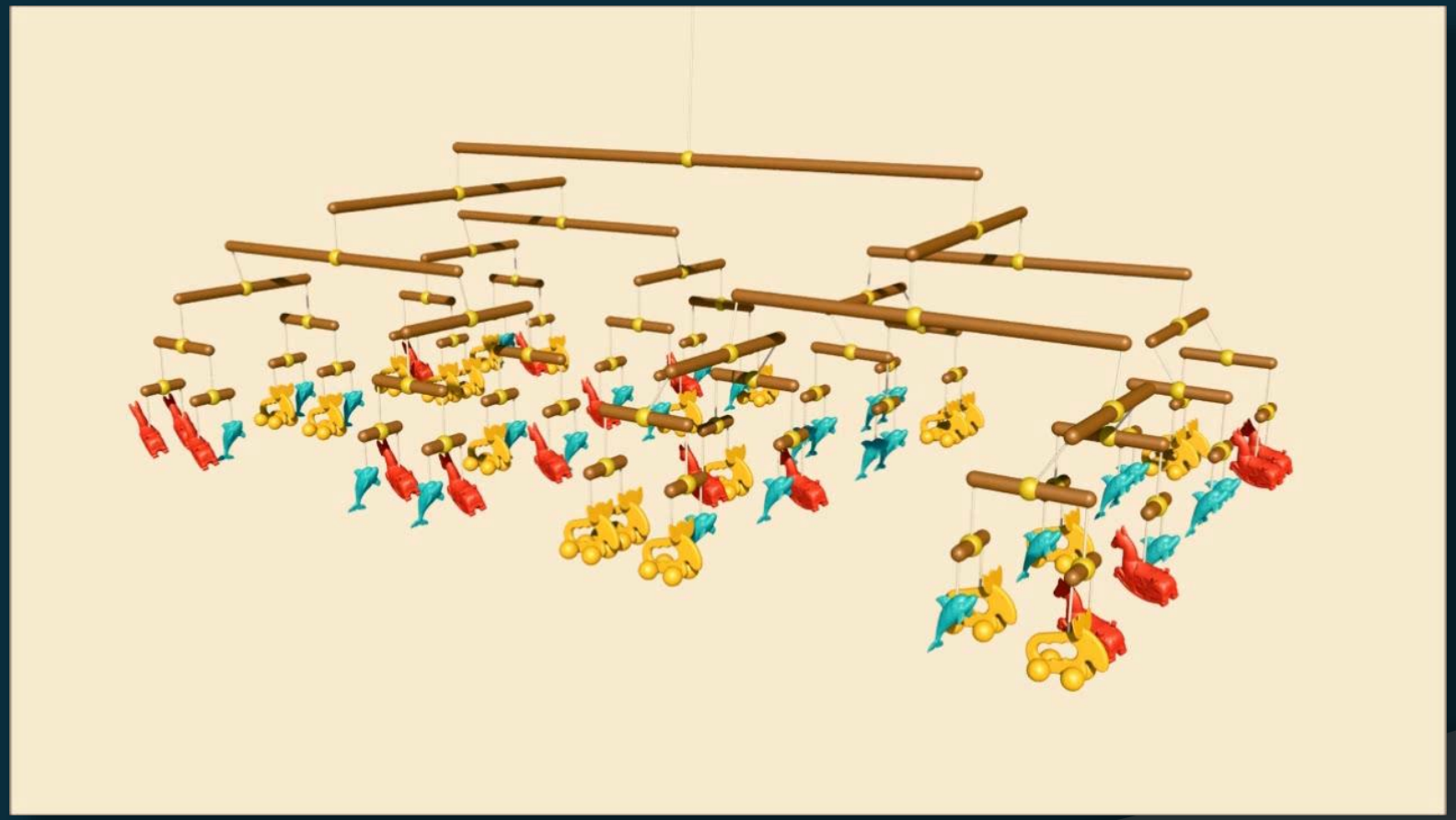
- Computation of Lagrange multiplier:

$$\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T\boldsymbol{\lambda} = -\mathbf{C}(\mathbf{x}, \boldsymbol{\vartheta})$$

- Correction vectors:

$$\left[\triangle\mathbf{x}^T, \triangle\boldsymbol{\vartheta}^T\right] = \mathbf{M}^{-1}\mathbf{J}^T\boldsymbol{\lambda}$$

Results



Topics

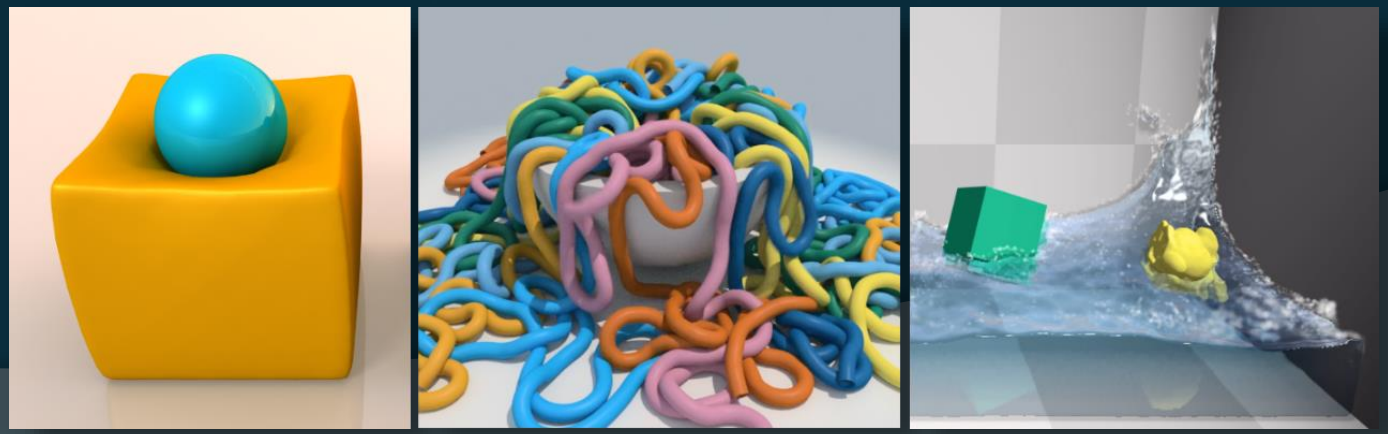
- Introduction
  Jan Bender
- Basic Position-Based Dynamics
  Matthias Müller
- Advanced Position-Based Dynamics
  Jan Bender
- Implementation & Analysis
  Miles Macklin
- Conclusion
  Jan Bender

Topics

- Introduction
  Jan Bender
- Basic Position-Based Dynamics
  Matthias Müller
- Advanced Position-Based Dynamics
  Jan Bender
- Implementation & Analysis
  Miles Macklin
- Conclusion
  Jan Bender

Conclusion

- Position-based methods:
  - are fast, stable and simple to implement,
  - provide a high level of control,
  - can simulate deformable solids (1D, 2D, 3D), multibody systems, fluids and granular materials,
  - support complex physical effects like anisotropy or plasticity and
  - provide visual plausibility.



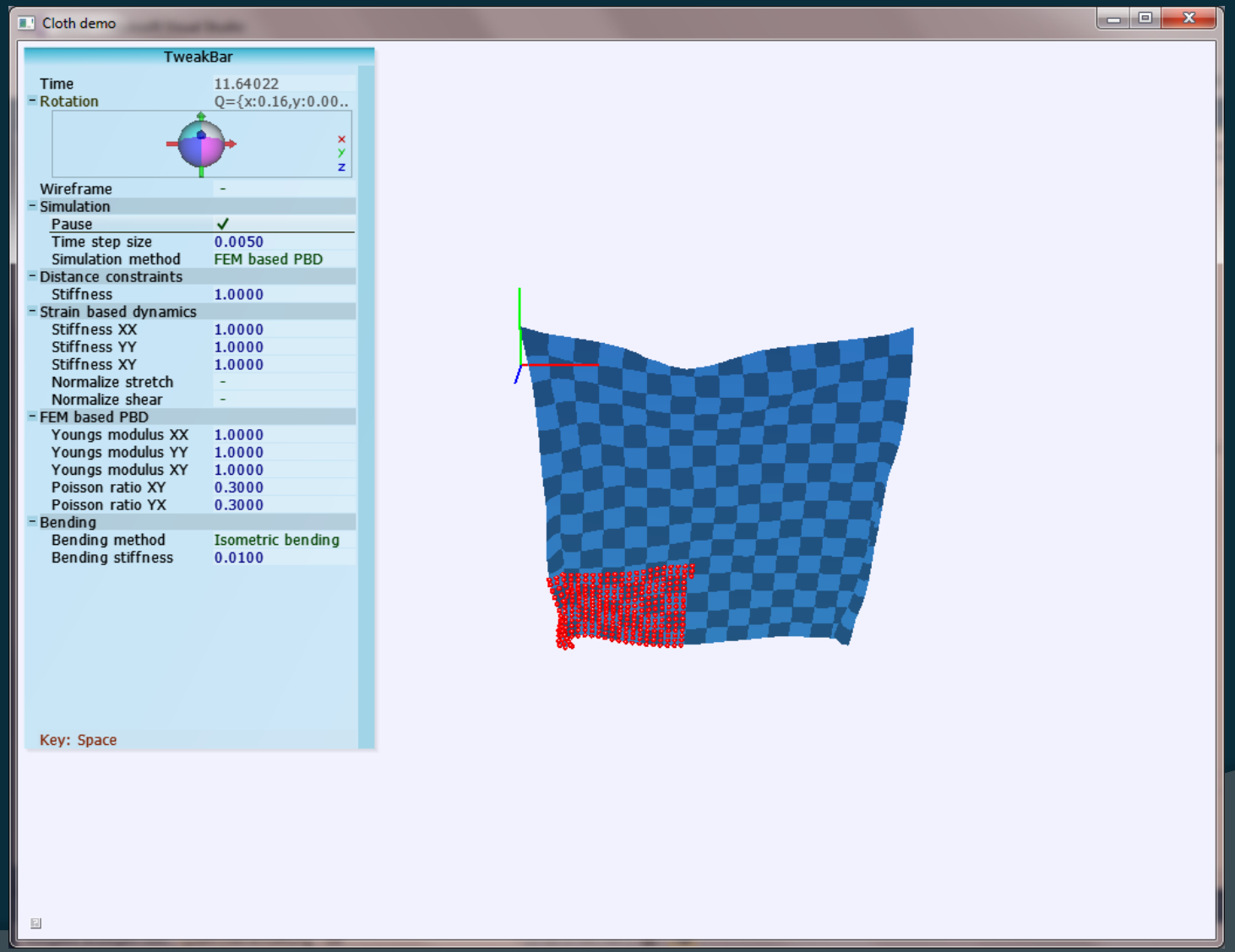
Future Work

- Decoupling of stiffness parameter, time step size and the number of solver iterations.
- Adaptive time stepping is an important topic for future work.
- Position-based dynamics is not convergent. Hence, the usage of adaptive meshes is another open problem.

PBD as Open-Source Library

- PBD is now available as open-source library
  - Already supports many constraints: point-point, point-edge, point-triangle and edge-edge distance constraints, dihedral bending constraint, isometric bending, volume constraint, shape matching, FEM-based PBD (2D & 3D), strain-based dynamics (2D & 3D).
  - Simple interface: just one class with static methods.
  - Library is free even for commercial applications.
  - Demos for usage.
  - Missing constraints will soon be added.

Demo


Cloth demo
TweakBar
Time 11.64022
Rotation Q={x:0.16,y:0.00..
x
y
z
Wireframe -
Simulation
Pause ✓
Time step size 0.0050
Simulation method FEM based PBD
Distance constraints
Stiffness 1.0000
Strain based dynamics
Stiffness XX 1.0000
Stiffness YY 1.0000
Stiffness XY 1.0000
Normalize stretch -
Normalize shear -
FEM based PBD
Youngs modulus XX 1.0000
Youngs modulus YY 1.0000
Youngs modulus XY 1.0000
Poisson ratio XY 0.3000
Poisson ratio YX 0.3000
Bending
Bending method Isometric bending
Bending stiffness 0.0100
Key: Space

Questions

